FACTORIZABLE Z(N) MODELS

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The factorizable S-matrix with $Z(N)$ symmetry is constructed. It is speculated that the field theory belonging to this S-matrix is related to the scaling limit of $Z(N)$ generalizations of the Ising model.

The two-dimensional Ising model has for a long time been the prototype of a system exhibiting a phase transition. After the complete solution for the correlation functions was obtained by McCoy et al. [1], it was shown that the field theory obtained by taking the scaling limit [2] of the Ising model has a factorizable [3] S-matrix. The knowledge of this S-matrix, which happens to be $S = -1$ [4], allows a reconstruction of the correlation functions [5].

A natural question arising in this context is how to obtain factorizable S-matrices belonging to theories with $Z(N)$ symmetry. These theories would be natural candidates describing the scaling limit of $Z(N)$ generalizations [6,7] of the Ising model. The values which the basic variables σ_i of these models assume are the N roots of unity. The σ_i's are coupled via global $Z(N)$ invariant short ranged interactions.

On the lattice the following identity holds:

$$\sigma_i^+ = \sigma_i^{N-1}. \tag{1}$$

In the scaling limit this equation becomes

$$\sigma^+(x) = \mathcal{N}[\sigma^{N-1}(x)], \tag{2}$$

where $\mathcal{N}$ stands for a suitable normal product prescription. Eq. (2) means that antiparticles are bound states of $N-1$ particles. In the chiral Gross–Neveu model such a property uniquely determined its exact S-matrix [8]. The same will be shown to occur here. As in the Ising model we expect the σ-field to describe particles with mass

$$m = \lim_{\substack{a\to 0\\ T\to T_c}} [(T - T_c)^\nu/a], \tag{3}$$

where a is the lattice spacing and ν is the critical exponent of the correlation length. Since antiparticles are bound states of particles, the reflection amplitude must vanish for a factorizable S-matrix. Hence, in standard notation [3] we have the following S-matrix elements:

$$\langle P_2 P_1 | S | P_1 P_2 \rangle = u(\theta_{12}), \tag{4a}$$

$$\langle \bar{P}_2 P_1 | S | P_1 \bar{P}_2 \rangle = t(\theta_{12}), \tag{4b}$$

where $P_i = m(\operatorname{ch}\theta_i, \operatorname{sh}\theta_i)$ and $\theta_{12} = \theta_1 - \theta_2$. Unitarity and crossing imply

$$u(\theta)u(-\theta) = 1, \quad t(\theta)t(-\theta) = 1, \tag{5a, b}$$

$$u(\theta) = t(\mathrm{i}\pi - \theta). \tag{5c}$$

If a pole, corresponding to a two-particle bound state, at $\theta_{12} = 2\pi \mathrm{i}/N$ is introduced in $u(\theta)$ the following n-particle bound state spectrum is generated [9]:

$$m_n = m \sin(\pi n/N)/\sin(\pi/N), \tag{6}$$

where now $m_{N-1} = m$. The solution of eq. (5) is given by [10]

$$u(\theta) = \mathrm{sh}\,\tfrac{1}{2}(\theta + 2\pi \mathrm{i}/N)/\mathrm{sh}\,\tfrac{1}{2}(\theta - 2\pi \mathrm{i}/N). \tag{7}$$

The fact that antiparticles are bound states of $N - 1$ particles requires the following consistency check: if in the N-particle scattering amplitude we project $N - 1$ particles onto the pole of mass m, we should reproduce the particle–antiparticle amplitude. This means that the following identity must hold:

$$\prod_n u(\theta + n\pi \mathrm{i}/N) = t(\theta) = u(\mathrm{i}\pi - \theta), \tag{8a}$$

where

$$\begin{aligned} n &= \pm 1, \pm 3, ..., \pm(N-2), \quad \text{for } N \text{ odd}, \\ &= 0, \pm 2, \pm 4, ..., \pm(N-2), \quad \text{for } N \text{ even}. \end{aligned} \tag{8b}$$

This is indeed true for $u(\theta)$ given by eq. (7), showing that eq. (7) gives the S-matrix of a $Z(N)$-invariant factorizable field theory.

A few remarks of general nature are now in order. Although our method of constructing the S-matrix has been devised for $N \geqslant 3$, it is gratifying to note that for $N = 2$ eq. (7) yields $S = -1$, as it should for the Ising model [4]. On the other hand, we expect a suitable $N \to \infty$ limit to describe the continuum limit of the XY model. Indeed, for $N \to \infty$ eq. (7) gives $S = 1$, in agreement with the fact the continuous O(2) σ-model is formally equivalent to a free massless theory. This equivalence, however, neglects the existence of vortices [11] which for $T > T_c$ play an essential role in building up short range correlations. Quantitatively the existence of spin waves and vortices is reflected by the appearance of two mass scales in the $N \to \infty$ limit of eq. (6). The lower mass m, which will be associated with spin waves, should vanish as $1/N$, whereas the higher mass $M \approx Nm$ should be identified with the inverse correlation length. At the same time we expect that the relevant operators of the continuous XY model should be composite operators corresponding to M. Although these qualitative remarks suggest that we are dealing with the continuous limit of $Z(N)$ models, a more detailed investigation requires the reconstruction of the correlation functions from the S-matrix along lines similar to the ones used in the Ising model [5]. Furthermore, the striking similarity [8] of the above models with the chiral Gross–Neveu model suggests a deeper link between them [‡1].

We plan to elaborate on those points in subsequent publications.

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‡1 The missing link can perhaps be constructed realizing that the product of order times disorder variables, which is a Fermi field in the Ising model, will obey generalized statistics in the Z(N) models. Fields with generalized statistics have, on the other hand, played a central role in our discussion of the chiral Gross–Neveu model.

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